Solving Systems of Differential Equations with Integrable Coefficients

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Approach

Consider a system of n differential equations of the form

$$\begin{cases} x_1' = f_{11}x_1 + f_{12}x_2 + \dots + f_{1n}x_n \\ x_2' = f_{21}x_1 + f_{22}x_2 + \dots + f_{2n}x_n \\ \vdots \\ x_n' = f_{n1}x_1 + f_{n2}x_2 + \dots + f_{nn}x_n \\ x_1(0) = x_{10}; x_2(0) = x_{20}; \dots; x_n(0) = x_{n0}. \end{cases}$$

where $n \in \mathbb{N}$, and $f_{ij} : [0, \infty) \to \mathbb{C}$ is an integrable function for all $i, j \in \{1, \dots, n\}$. Let $M_{n\times n}(\mathbb{C})$ denote the set of n by n matrices with complex entries. If we define the matrix of functions

$$A := \begin{pmatrix} f_{11} & \dots & f_{1n} \\ \vdots & \dots & \vdots \\ f_{n1} & \dots & f_{nn} \end{pmatrix},$$

and $\boldsymbol{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))$, then $\boldsymbol{x}'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))$. The system may then be rewritten as an IVP that we aim to solve:

$$\begin{cases} \boldsymbol{x}'(t) = A(t)\boldsymbol{x}(t) \\ \boldsymbol{x}(0) = \boldsymbol{x_0} \end{cases}$$

Introduction

Systems of homogeneous, linear first-order ODEs with integrable coefficients have many applications in the scientific world. The wide range of dynamical systems modeled by such equations continues to push scientists in a wide variety of fields to search for friendly techniques to find solutions.

Transformation $\mathcal{F}$

For an integrable $A : [0, \infty) \to M_{n\times n}(\mathbb{C})$, we define the transformation $\mathcal{F}$ as follows for each integrable $Y : [0, \infty) \to M_{n\times n}(\mathbb{C})$ and all $t \geq 0$:

$$\mathcal{F}(\boldsymbol{y})(t) := \int_0^t A(\tau)\boldsymbol{y}(\tau)d\tau$$

for each integrable $\boldsymbol{y} : [0, \infty) \to \mathbb{C}^n$ and for all $t \geq 0$. Notice that we have that $\boldsymbol{x}(t) = \boldsymbol{x_0} + \mathcal{F}(\boldsymbol{x})(t)$ and that $\mathcal{F}$ is linear. Also notice that in special cases such as $\mathcal{F}$, for any constant vector $\boldsymbol{c} \in \mathbb{C}^n$, $\mathcal{F}(Y\boldsymbol{c})(t) = \int_0^t A(\tau)Y(\tau)\boldsymbol{c}d\tau = \mathcal{F}(Y)(t)c$ as we expect.

Transformation limit for $\mathcal{F}$ for approximation

Let $A : [0, \infty) \to M_{n\times n}(\mathbb{C})$ be integrable.

The transformation $\mathcal{F}$ is linear, and for all $t_0 > 0$, $n \geq 0$ for all integrable $\boldsymbol{x} : [0, \infty) \to \mathbb{C}^n$, we have

$$\sup_{t\in[0,t_0]} \|\mathcal{F}^k(\boldsymbol{x})(t)\| \leq \frac{(\sup_{t\in[0,t_0]} \|A(t)\|t_0)^k}{k!} \sup_{t\in[0,t_0]} \|\boldsymbol{x}(t)\|$$

Now that we have this lemma, we finally have the tools to show that an infinite k-fold composition of $\mathcal{F}$ as $k \to \infty$ converges.

Solution form to the IVP

Let $A : [0, \infty) \to M_{n\times n}(\mathbb{C})$ be integrable, and $\boldsymbol{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))$, with $\boldsymbol{x}'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))$, then from we have

$$\boldsymbol{x}(t) = \boldsymbol{x_0} + \int_0^t A(\tau)\boldsymbol{x}(\tau)d\tau$$

Now we have an idea for formulating a solution for $\boldsymbol{x}$ in general. However, the integral $\int_0^t A(\tau)\boldsymbol{x}(\tau)d\tau$ is not computable in general; so, we search for a new approach.

Series B

It follows easily from induction that $\boldsymbol{x}(t) = \sum_{k=0}^{n} \mathcal{F}^k(I)(t)\boldsymbol{x_0} + \mathcal{F}^{n+1}(\boldsymbol{x})(t)$ for all $n > 0$, where I is the n by n identity matrix viewed as a function and $\mathcal{F}^k$ is the k-fold composition of $\mathcal{F}$.

For an integrable $A : [0, \infty) \to M_{n\times n}(\mathbb{C})$, define $\mathcal{F}$ as above. Then if it exists, we define a function $B : [0, \infty) \to M_{n\times n}(\mathbb{C})$ as

$$B(t) := \sum_{k=0}^{\infty} \mathcal{F}^k(I)(t)$$

for $t \in [0, \infty)$.

Series B as the unique solution to the IVP

Let $A : [0, \infty) \to M_{n\times n}(\mathbb{C})$ be integrable. Then B exists and

$$B(t)\boldsymbol{x}(0) = \sum_{k=0}^{\infty} \mathcal{F}^k(I)(t)\boldsymbol{x}(0)$$

is the unique solution to the IVP $\boldsymbol{x}'(t) = A(t)\boldsymbol{x}(t)$ for first differentiable $\boldsymbol{x} : [0, \infty) \to \mathbb{C}^n$ with initial condition $\boldsymbol{x}(0)$. A sketch of the proof is provided:

Proof: Series B as the unique solution to the IVP

Let $\boldsymbol{w}(t) : [0, \infty) \to \mathbb{C}^n$ be a solution to the IVP for $t \in [0, \infty)$. Then, we have $\boldsymbol{w}(t) - \boldsymbol{x}(0) = \int_0^t A(\tau)\boldsymbol{w}(\tau)d\tau$.

Using our definition of $\mathcal{F}$, we have

$$\boldsymbol{w}(t) = x(0) + \int_0^t A(\tau)\boldsymbol{u}(\tau)d\tau = \mathcal{F}^0(I)(t)x(0) + \mathcal{F}(\boldsymbol{w})(t)$$

Showing $\boldsymbol{w}(t) = \sum_{s=0}^{0} \mathcal{F}^s(I)(t)\boldsymbol{x}(0) + \mathcal{F}^{0+1}(\boldsymbol{w})(t)$ and $\boldsymbol{w}(t) = \sum_{s=0}^{k+1} \mathcal{F}^s(I)(t)\boldsymbol{x}(0) + \mathcal{F}^{k+2}(\boldsymbol{w})(t)$ completes the proof by induction.

Finally, $\|\boldsymbol{w}(t) - B(t)\boldsymbol{x}(0)\| = 0$ proves uniqueness of our solution.

Commutative Approximation Theorem

Let $A : [0, \infty) \to M_{n\times n}(\mathbb{C})$ be an integrable function. Also define $C : [0, \infty) \to M_{n\times n}(\mathbb{C})$ as $C(t) = \mathcal{F}(I)(t)A(t) - A(t)\mathcal{F}(I)(t)$. Then for all $t_0 \geq 0$,

$$\sup_{t\in[0,t_0]} \|e^{\int_0^t A(\tau)d\tau} - B(t)\| \leq \frac{\sup_{t\in[0,t_0]} \|C(t)\|t_0}{2} e^{\sup_{t\in[0,t_0]} \|A(t)\|t_0}$$

Case: A is a matrix of constant entries

- Notice that if A is a constant, then $B(t) = \sum_{k=0}^{\infty} \mathcal{F}^k(I)(t) = \sum_{k=0}^{\infty} \frac{(At)^t}{k!}$, similar to the terms of the series representation of e, namely $e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$.
- Let $A \in M_{n\times n}(\mathbb{C})$. Then if it exists,

$$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

- e^A does exist for all $A \in M_{n\times n}(\mathbb{C})$ and has many properties similar to e defined over the real or complex numbers.
- Suppose we seek to solve the IVP $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$ for first differentiable $\boldsymbol{x} : [0, \infty) \to \mathbb{C}^n$ with initial condition $\boldsymbol{x}(0)$
  - Recall $\sum_{k=0}^{\infty} \mathcal{F}^k(I)(t)\boldsymbol{x}(0)$ is the unique solution to the IVP where $\mathcal{F}(X)(t) = \int_0^t AX(\tau)d\tau$
  - $\sum_{k=0}^{\infty} \mathcal{F}^k(I)(t)\boldsymbol{x}(0) = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}\boldsymbol{x}(0) = e^{tA}\boldsymbol{x}(0)$
  - The unique solution to this IVP is then

$$e^{tA}\boldsymbol{x}(0)$$

Constant Case Example

Many lakes are part of a complex system of interconnected bodies of water. If one or more of these bodies of water is polluted, then it comes as no surprise that the pollution will spread throughout the system.

- Assuming, for example, that the original amount of pollution (in pounds) in the Lakes is $1000, 1100, 900, 0,$ and 300 respectively, $r_i = 1$ for $i \in \{1, 2, \dots, 5\}$, and $V = \{2900, 1180, 850, 116, 393\}$ (measured in cubic miles) respectively
- Then we may develop a system that models the evolution of pollution throughout the lake system:

$$(IVP) \quad \begin{cases} x'(t) = Ax(t) \\ x(0) = x_0 \end{cases}$$

with $x_0 = (1000, 1100, 900, 0, 300)^{\mathsf{T}}$

- We may then write this system in matrix form with each entry $a_{ij} \in A$.
- We are then able to compute $e^{tA}x_0 = \mathcal{L}^{-1}[(\lambda I - A)^{-1})(t)x_0 =$

$$\begin{pmatrix} e^{-\frac{10t}{29}} & 0 & 0 & 0 & 0 \\ \frac{-649e^{-\frac{165t}{59}-\frac{10t}{29}}}{4195} & e^{-\frac{165t}{59}} & \frac{-1298e^{-\frac{165t}{59}-\frac{18t}{17}}}{1743} & 0 & 0 \\ 0 & 0 & e^{-\frac{18t}{17}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{393}{145}e^{-\frac{500t}{131}+\frac{500t}{131}} & e^{-\frac{500t}{131}} \end{pmatrix}$$

which has the modeled solution: